Chebyshev polynomials for three-dimensional algebra

C_3-case

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Motivation

Multivariate Chebyshev polynomials constitute a full orthogonal and normalized basis for $\mathcal{L}^2$ functions on deformed triangles, tetrahedra and higher dimensional simplexes. They are applied in quantum physics Borzov-Damaskinskiy-2012 and numerical analysis Gehlen-Roan.
In 1974, Koornwinder, extended the idea of converting characters of A_1-modules into a set of Chebyshev orthogonal polynomials by constructing and studying two-variable polynomials corresponding to the set of A_2-highest weight modules $L^{\mu}_{A_2}$ and found that their domain of definition is a curved triangle region bounded by the so-called Steiner hypocycloid. A systematic study of multivariate A_n-Chebyshev polynomials was performed in 1991 by R. J. Beerends the corresponding measure for $\mathcal{L}^2_{A_n}$ was constructed.

Motivation

H. Z. Munthe-Kaas and B. Ryland considered the role of multivariate Chebyshev T-polynomials of the first type in Fourier analysis and approximation theory. Connections between multivariate
Most of useful properties of Chebyshev polynomials arise from their substantial connection with the group theory. Taking this into account we consider characters $\mathrm{ch}\left(L_{\mathfrak{g}}^{\mu}\right)$ of irreducible highest weight modules $L_{\mathfrak{g}}^{\mu}$ to contain most of information about the initial symmetry and polynomials of the second type $U_{\mathfrak{g}}^{\mu}(X)$ originating from specialized characters $\mathrm{ch}\left(L_{\mathfrak{g}}^{\mu;a}\right)$ to be of highest interest.

Motivation

In Lyakhovsky-Uvarov-2012 a simple expression was found for the measure that realizes the scalar product in $\mathcal{L}^2_{\mathfrak{g}}(X)$ providing the orthogonality and normalization for the set $\{U^{\mu}_{\mathfrak{g}}(X)\}$.
We construct multivariate polynomials $U^{\mu}_{\mathfrak{g}}(X)$ using singular elements $\Psi^{(\mu)}_{\mathfrak{g}}$ properties.

Some definitions and relations

$\mathfrak{g}$ – the semi-simple Lie algebra
r – rank of $\mathfrak{g}$
$\Delta_{\mathfrak{g}}$ – root system for a semi-simple Lie algebra $\mathfrak{g}$
$\Delta_{\mathfrak{g}}^{+}$ – positive root system
$W_{\mathfrak{g}}$ – Weyl group for system $\Delta_{\mathfrak{g}}$ with the sign function
$\epsilon(w) := \det(w) = (-1)^{length(w)}$, $w \in W_{\mathfrak{g}}$
$C^{(0)}$ – fundamental Weyl chamber and $\overline{C^{(0)}}$ – its closure;
α_i – i-th simple root for $\mathfrak{g}$, $i = 0, \ldots, r$;
$\theta = \sum_{j=1}^{r} \frac{2}{|\alpha_j|^2} q_j \alpha_j = \sum_{i=1}^{r} q_i \widehat{\alpha}_i$ – the highest root with comarks q_i;
ω_i – i-th fundamental weight for $\mathfrak{g}$;
$\rho_{\mathfrak{g}}$ – the Weyl vector;

Some definitions and relations

L^{μ} – an integrable $\mathfrak{g}$-module with highest weight μ;
$L_{\mathfrak{g}}^{\omega_i}$ – i-th fundamental module;
$P_{\mathfrak{g}}$ – weight lattice, $P_{\mathfrak{g}}^{++}$ – dominant weight lattice, $\mathcal{LP}_{\mathfrak{g}}$ – weight spare;
$\Psi_{\mathfrak{g}}^{(\mu)} := \sum_{w \in W} \epsilon(w) e^{w \circ (\mu+\rho)-\rho}$ – singular element for a $\mathfrak{g}$-module L^{μ};
$\Phi_{\mathfrak{g}}^{(\mu)} := \sum_{w \in W} \epsilon(w) e^{w \circ (\mu+\rho)}$ – nonshifted singular element;
$R_{\mathfrak{g}} := \prod_{\alpha \in \Delta_{\mathfrak{g}}^{+}} (1 - e^{-\alpha}) = \Psi^{(0)}$ – the Weyl denominator.

Chebyshev polynomials

We represent the multivariate Chebyshev polynomials which constitute a full orthogonal and normalized basis in the r-space, $\mathbf{R}^r$, in terms of x. Where $a(x) := 2\pi i \sum^r \alpha^j x_j$ there are the spin multiplicity the torus. We consider characters of $\mathrm{ch}\,(L^{\mu}_{\mathfrak{g}})$ polynomials of the second type $U^{\mu}_{\mathfrak{g}}(X)$

$$U^{\mu}_{\mathfrak{g}}(x) := \mathrm{ch}\left(L^{\mu;a(x)}\right) = \sum_{\nu \in P} c^{(\mu)}_{\nu} \exp\left(\nu, a(x)\right). \tag{1}$$

Here $c^{(\mu)}_{\nu}$ are weight multiplicities for L^{μ} , $\nu = \sum^r \nu^j \omega_j$. The conjugated polynomial

$$\overline{U^{\xi}(X)} = \sum_{\nu \in \mathcal{N}^{\mu^*}} c^{(\mu)}_{\nu} \exp\left(2\pi i \sum_{j=1}^{r} \nu_j x^j\right) = U^{\mu^*}(X)$$

corresponds to the contragradient module $L^{\mu^*} = (L^{\mu})^{\mathsf{T}-1}$.

Results

Fundamental modules L^{ω_j}, ($j = 1, \ldots, r$),

$$\left\{X^j\right\} := \left\{U_{\mathfrak{g}}^{\omega_j}(x)\right\}_{j=1,\ldots,r} \tag{2}$$

give us the set of independent variables $X \longleftrightarrow x$.
As far as Chebyshev functions (and correspondingly Chebyshev polynomials) are defined as specialized characters for the full set of irreducible finite dimensional modules $L_{\mathfrak{g}}$ the system $\{U_{\mathfrak{g}}^{\mu}\}_{|\mu \in P^{++}}$ forms a basis in $\mathcal{L}^2(\mathcal{F}_{\mathfrak{g}})$. This gives us explicit expressions for characters $\{U^{\mu}\}$ composing the set of polynomials $\left\{ U_{\mathfrak{g}}^{\mu}{}_{|\mu \in P^{++}} \right\}$.

Results

We are to fix the domain of definition $\mathcal{F}_{\mathfrak{g}}$ (in $\mathbf{R}^r$) for functions in $\mathcal{L}^2(\mathcal{F}_{\mathfrak{g}})$ of variables $\{X_j\}$ to find there an appropriate measure μ and thus to define the normalized invariant scalar product for functions on $\mathcal{F}_{\mathfrak{g}}$. In variables x the sequence $\{q_j\}$ forms the set comarks for $\mathfrak{g}$. The fundamental domain $F_{\mathfrak{g}} \subset \overline{C_{\mathfrak{g}}^{(0)}} \subset \mathcal{L}P_{\mathfrak{g}}$ contains the points ν of the dominant Weyl chamber (for which $(\nu, \vartheta_{\mathfrak{g}}) \leq 1$):

$$F_{\mathfrak{g}} = \mathrm{Conv}\left\{0, \frac{\omega_1}{q_1}, \frac{\omega_2}{q_2}, \ldots, \frac{\omega_r}{q_r}\right\}.$$

This means that $F_{\mathfrak{g}}$ is the closure of the "alcove". In angle variables the fundamental domain depends on a choice of the specialization vector $a(x)$. The set $\left\{\Phi^{\mu;a(x)}\right\}_{\mu \in P^{++}}$ of non-shifted singular functions on $F_{\mathfrak{g}}(x)$ is orthogonal and normalized [**?**].

Results

Here $|F_{\mathfrak{g}}(x)|$ and $|W_{\mathfrak{g}}|$ are volumes of the fundamental domain and the group $W_{\mathfrak{g}}$.

$$\frac{1}{|F_{\mathfrak{g}}(x)|\,|W_{\mathfrak{g}}|}\int_{F_{\mathfrak{g}}(x)}\Phi^{\mu;\alpha(x)}\overline{\Phi^{\lambda;\alpha(x)}}dx^1\ldots dx^r=\delta^{\mu\lambda}. \tag{3}$$

From the famous Weyl formula we attribute

$$\mathrm{ch}\left(L^{\mu}\right)=\frac{\sum_{w\in W}\epsilon(w)e^{w\circ(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta^+}\left(1-e^{-\alpha}\right)}=\frac{\psi(\mu)}{\psi(0)}, \tag{4}$$

formal character of L^{μ} and pass from $X\longleftrightarrow x$:

$$\begin{aligned}&\frac{1}{|F_{\mathfrak{g}}(x)|\,|W_{\mathfrak{g}}|}\Phi^{\mu;\alpha(x)}\overline{\Phi^{\lambda;\alpha(x)}}dx^1\ldots dx^r\\ =\;&\mathrm{ch}\left(L^{\mu;\alpha(x)}\right)\mathrm{ch}\left(L^{\lambda^*;\alpha(x)}\right)\frac{1}{|F_{\mathfrak{g}}(x)|\,|W_{\mathfrak{g}}|}\Phi^{0;\alpha(x)}\overline{\Phi^{0;\alpha(x)}}\frac{1}{\left|\frac{\partial X_i}{\partial x_j}\right|}dX^1\ldots dX^r\end{aligned} \tag{5}$$

Results

$$\frac{\Phi^{0;a(x(X))}\overline{\Phi^{0;a(x(X))}}}{|F_{\mathfrak{g}}(x(X))|\,|W_{\mathfrak{g}}|\left|\frac{\partial X_i}{\partial x_j}\right|}dX^1\ldots dX^n=\gamma_{\mathfrak{g}}(X)\,dX^1\ldots dX^r. \tag{6}$$

In Lyakhovsky-Uvarov-2012 it was demonstrated that for any semi-simple Lie algebra $\mathfrak{g}$ there exists a unique function $\gamma_{\mathfrak{g}}$ that can be easily expressed in terms of singular element $\Phi_{\mathfrak{g}}^{0;a(x(X))}$:

$$\gamma_{\mathfrak{g}}(X)=\frac{\overline{\Phi_{\mathfrak{g}}^{0;a(x(X))}}}{(2\pi i)^r\,|F_{\mathfrak{g}}(x(X))|\,|W_{\mathfrak{g}}|}. \tag{7}$$

Results

The factor $\overline{\Phi^{0;a(x(X))}}$ in $\gamma(X)$ cannot be decomposed into a linear combination of characters. On the contrary, the element $\left(\Phi^{0;a(x(X))}\right)^2$ is W-invariant and can be expressed in terms of $U^{\nu}(X)$. The only information that we need to perform such a decomposition is a sufficient set of polynomials $U^{\nu}(X)$. It is sufficient to decompose the W-invariant (described by a multi-index $I(p)$ as well as in the set of Chebyshev-like polynomials) $\left(\Phi^{0;\alpha(X)}\right)^2 = b_{I(p)}X^{I(p)}$ and introduce the factors $\Phi^{0;\alpha(X)} = \sqrt{b_{I(p)}X^{I(p)}}$, $q_{\mathfrak{g}}$ and $(-1)^{\mathrm{maxlength(W)}}$. The second factor is the number of alcoves in the domain $F(x)$. The third factor depends on the properties of singular elements with respect to conjugation,

$$\gamma_{\mathfrak{g}}(X) = \sqrt{b_{I(p)}X^{I(p)}}\ \frac{q_{\mathfrak{g}}(-1)^{\mathrm{maxlength(W)}}}{(2\pi i)^r |F_{\mathfrak{g}}||W_{\mathfrak{g}}|}. \tag{8}$$

Polynomials

Consider the tensor product $\left(L_{\mathfrak{g}}^{\omega_1}\right)^{\otimes p_1} \otimes \left(L_{\mathfrak{g}}^{\omega_2}\right)^{\otimes p_2} \otimes \ldots \otimes \left(L_{\mathfrak{g}}^{\omega_r}\right)^{\otimes p_r}$ with $\{p_j\} \in (\mathbf{Z}_{\geq 0})^r$ and its specialization $(X^1)^{p_1} \ldots (X^r)^{p_r}$. Introduce a partial ordering in $P_{\mathfrak{g}}^{++}$ by projecting the vector $\sum_i^r p_i \omega_i$ to an appropriate test vector. Collecting the terms with a fixed dominant weight $\mu \in P^{++}$ we obtain a positive integer coefficients $m_{\nu}^{l(p)}$. These coefficients form a triangular matrix $M_{\mathfrak{g}}$ that has units in its main diagonal.

We attribute the character of $\left(L_{\mathfrak{g}}^{\omega_1}\right)^{\otimes p_1} \otimes \left(L_{\mathfrak{g}}^{\omega_2}\right)^{\otimes p_2} \otimes \ldots \otimes \left(L_{\mathfrak{g}}^{\omega_r}\right)^{\otimes p_r}$ with $l(p) = \sum_{i=1}^r p_i w_i$.

Polynomials

Lemma

We are to fix the domain of definition for functions in of variables $\{X_j\}$ *to find there an appropriate measure* μ *and thus to define*

$$\begin{aligned} & \int_0^1 \dots \int_0^1 \left(L_{\mathfrak{g}}^{\omega_1}\right)^{\otimes p_1} \dots \left(L_{\mathfrak{g}}{}^{\omega_r}\right)^{\otimes p_r} \Phi^{(0)} e^{-\rho} |_{\overline{C^{(0)}}} e^{\sum_j^r l_j \omega_j} dx^1 \dots dx^r \\ = \quad & m^{\left(\sum_i^r p_i \omega_i\right)\left(\sum_j^r l_j \omega_j\right)}, \end{aligned}$$

(the l_j *are negative and are* $\omega_j(x) := 2\pi i x^j$ *).*

Polynomials

We attribute the character of $\left(L_{\mathfrak{g}}^{\omega_1}\right)^{\otimes p_1} \otimes \left(L_{\mathfrak{g}}^{\omega_2}\right)^{\otimes p_2} \otimes \ldots \otimes \left(L_{\mathfrak{g}}^{\omega_r}\right)^{\otimes p_r}$ with

$$\begin{aligned}\left(L_{\mathfrak{g}}^{\omega_1}\right)^{\otimes p_1} \otimes \ldots \otimes \left(L_{\mathfrak{g}}^{\omega_r}\right)^{\otimes p_r} &= \sum_{\mu \in P_g} m^{\left(\sum_{i=1}^{r} p_i w_i\right)} \frac{\sum_{w \in W} \epsilon(w) e^{w \circ (\mu+\rho)-\rho}}{\prod_{\alpha \in \Delta^+}\left(1-e^{-\alpha}\right)} = \\ &= \sum_{\mu \in P_g} m^{l(p)} \frac{\Psi(\mu)}{\Psi(0)},\end{aligned}$$

or producting for Weyl vector

$$\left(L_{\mathfrak{g}}^{\omega_1}\right)^{\otimes p_1} \otimes \ldots \otimes \left(L_{\mathfrak{g}}^{\omega_r}\right)^{\otimes p_r} \Phi^{(0)} e^{-\rho} = \sum_{\mu \in P_g} m^{l(p)} \Psi(\mu),$$

Polynomials

and transfering to the closure of Weyl chamber

$$\left(L_{\mathfrak{g}}^{\omega_1}\right)^{\otimes p_1} \dots \left(L_{\mathfrak{g}}^{\omega_r}\right)^{\otimes p_r} \Phi^{(0)} e^{-\rho}|_{\overline{C^{(0)}}} = \sum_{\mu \in P_g} m^{\left(\sum_i^r p_i \omega_i\right)} \sum_{w \in W} e^{w \circ (\mu+\rho)-\rho}|_{\overline{C^{(0)}}}.$$

The normalized invariant scalar product is ...

$$\left\{1, e^{-\omega_1}, e^{-\omega_2}, \dots, e^{-\omega_r}, e^{-2\omega_1}, e^{-2\omega_2}, \dots, e^{-2\omega_r}, e^{-\omega_1-\omega_2}, e^{-2\omega_1-\omega_2}, \dots\right\}$$

and after integration we receive the concrete number

$$\begin{aligned} & \int_0^1 \dots \int_0^1 \left(L_{\mathfrak{g}}^{\omega_1}\right)^{\otimes p_1} \dots \left(L_{\mathfrak{g}}{}^{\omega_r}\right)^{\otimes p_r} \Phi^{(0)} e^{-\rho}|_{\overline{C^{(0)}}} e^{\sum_i^r l_i \omega_i} dx^1 \dots dx^r \\ = \quad & m^{\left(\sum_i^r p_i \omega_i\right)\left(\sum_j^r l_j \omega_j\right)}. \end{aligned}$$

Lemma reminds of Eric Verlinde formular.

Example. Chebyshev polynomials for B_2.

Fundamental weights for the algebra B_2 are

$$\begin{aligned} L_{\mathfrak{g}}^{\omega_1} &:= e^{\omega_1} + e^{-\omega_1} + e^{(\omega_1 - 2\omega_2)} + e^{(-\omega_1 + 2\omega_2)} + 1; \\ L_{\mathfrak{g}}^{\omega_2} &:= e^{\omega_2} + e^{-\omega_2} + e^{(\omega_1 - \omega_2)} + e^{(-\omega_1 + \omega_2)}. \end{aligned}$$

The Weyl vector here is

$$\begin{aligned} \rho_{\mathfrak{g}} = \Phi_{\mathfrak{g}}^{(0)} &:= e^{(\omega_1+\omega_2)} - e^{(2\omega_1-\omega_2)} + e^{(2\omega_1-3\omega_2)} - e^{(\omega_1-3\omega_2)} + \\ &+ e^{(-\omega_1-\omega_2)} + e^{(-2\omega_1+\omega_2)} - e^{(-\omega_1+3\omega_2)} + e^{(-2\omega_1+3\omega_2)}. \end{aligned}$$

The transform $\omega_j = 2\pi i x^j$.

Example. Chebyshev polynomials for B_2.

The fundamental weights in geometrical variables

$$\begin{aligned} L_{\mathfrak{g}}^{\omega_1}(x) &= e^{2\pi i x_1} + e^{2\pi i(-x_1)} + e^{2\pi i(x_1-2x_2)} + e^{2\pi i(-x_1+2x_2)} + 1, \\ L_{\mathfrak{g}}^{\omega_2}(x) &= e^{2\pi i x_2} + e^{2\pi i(-x_2)} + e^{2\pi i(x_1-x_2)} + e^{2\pi i(-x_1+x_2)}. \end{aligned}$$

The Weyl vector is

$$\begin{aligned} \Phi_{\mathfrak{g}}^{(0)}(x) &= e^{2\pi i(x_1+x_2)} - e^{2\pi i(2x_1-x_2)} + e^{2\pi i(2x_1-3x_2)} - e^{2\pi i(x_1-3x_2)} + e^{2\pi i(-x_1-x_2)} + e^{2\pi i(-2x_1+x_2)} - \\ &\quad - e^{2\pi i(-x_1+3x_2)} + e^{2\pi i(-2x_1+3x_2)} \\ \Psi_{\mathfrak{g}}^{(0)}(x) &= \Phi_{\mathfrak{g}}^{(0)}(x)\, e^{2\pi i(-x_1-x_2)}. \end{aligned}$$

Using the obtained Weyl vector we write down the important decomposition explicitly mutiplying by the arbitrary members (no matter what) "$\left(L_{\mathfrak{g}}^{\omega_2}(x)\right)^3$"

Example. Chebyshev polynomials for B_2.

$$\Phi_{\mathfrak{g}}^{03}(x) := \left(L_{\mathfrak{g}}^{\omega_2}(x)\right)^3 \Psi_{\mathfrak{g}}^{(0)}(x);$$

$$\Phi_{\mathfrak{g}}^{03}(x) := \left(e^{2\pi i x_2} + e^{2\pi i(-x_2)} + e^{2\pi i(x_1-x_2)} + e^{2\pi i(-x_1+x_2)}\right)^3 \times$$
$$\left(e^{2\pi i(x_1+x_2)} - e^{2\pi i(2x_1-x_2)} + e^{2\pi i(2x_1-3x_2)} - e^{2\pi i(x_1-3x_2)} + e^{2\pi i(-x_1-x_2)} + \right.$$
$$\left. + e^{2\pi i(-2x_1+x_2)} - e^{2\pi i(-x_1+3x_2)} + e^{2\pi i(-2x_1+3x_2)}\right) e^{2\pi i(-x_1-x_2)}$$

The function returns the following results.
Being multiplyed by '1' it gives

$$\int_0^1 \int_0^1 \left(L_{\mathfrak{g}}^{\omega_2}(x)\right)^3 \Psi_{\mathfrak{g}}^{(0)}(x) dx_1 dx_2 = 0.$$

Being multiplyed by '$e^{2\pi i(-x_2)}$' it will be

Example. Chebyshev polynomials for B_2.

$$\int_0^1 \int_0^1 \left(L_{\mathfrak{g}}^{\omega_2}(x)\right)^3 \psi_{\mathfrak{g}}^{(0)}(x) e^{2\pi i(-x_2)} dx_1 dx_2 = 3,$$

and " $e^{2\pi i(-x_1)}, e^{2\pi i(2x_2)}$" the following result

$$\begin{aligned}\int_0^1 \int_0^1 \left(L_{\mathfrak{g}}^{\omega_2}(x)\right)^3 \psi_{\mathfrak{g}}^{(0)}(x) e^{2\pi i(-x_1)} dx_1 dx_2 &= 0,\\ \int_0^1 \int_0^1 \left(L_{\mathfrak{g}}^{\omega_2}(x)\right)^3 \psi_{\mathfrak{g}}^{(0)}(x) e^{2\pi i(-2x_2)} dx_1 dx_2 &= 0.\end{aligned}$$

Then being multiplyed by "$e^{2\pi i(-x_1-x_2)}$" –

$$\int_0^1 \int_0^1 \left(L_{\mathfrak{g}}^{\omega_2}(x)\right)^3 \psi_{\mathfrak{g}}^{(0)}(x) e^{2\pi i(-x_1-x_2)} dx_1 dx_2 = 2,$$

and

$$\int_0^1 \int_0^1 \left(L_{\mathfrak{g}}^{\omega_2}(x)\right)^3 \psi_{\mathfrak{g}}^{(0)}(x) e^{2\pi i(-2x_1)} dx_1 dx_2 = 0,$$

Example. Chebyshev polynomials for B_2.

and

$$\int_0^1 \int_0^1 \left(L_{\mathfrak{g}}^{\omega_2}(x)\right)^3 \Psi_{\mathfrak{g}}^{(0)}(x) e^{2\pi i(-3x_2)} dx_1 dx_2 = 1.$$

Next coefficients " $e^{2\pi i(x_1+2x_2)}, e^{2\pi i(2x_1+x_2)}, e^{8\pi i x_2}, \ldots$" will give zero. Thus we obtained the series,

$$0, 3, 0, 0, 2, 0, 1, 0, 0, 0, \ldots$$

This is all Chebyshev polynomials for B_2 and $\left(L_{\mathfrak{g}}^{\omega_2}(x)\right)^3$.

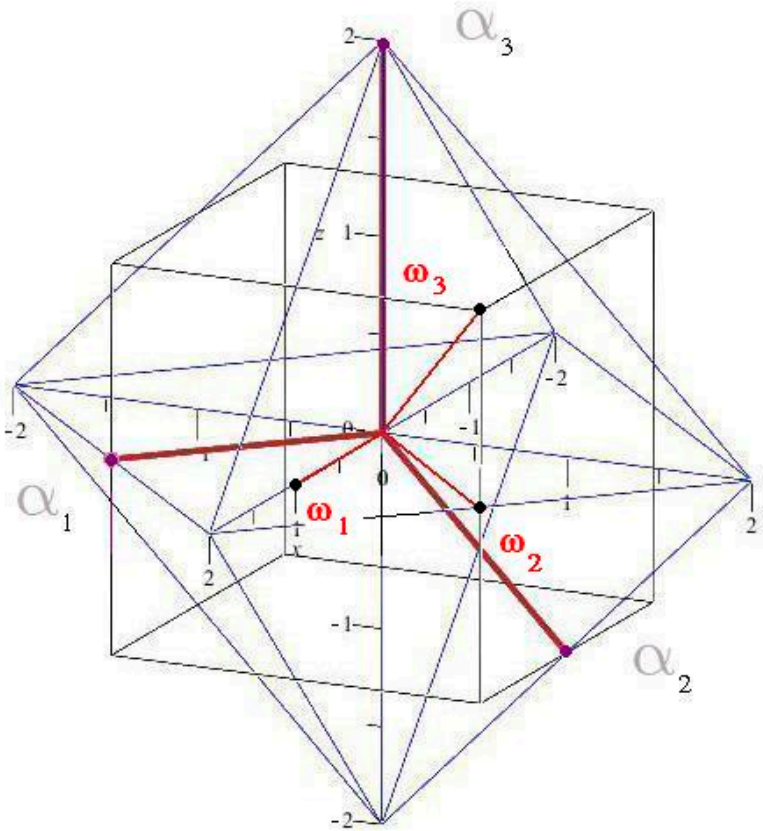
C_3

Simple roots and fundamental weights are

$$
\begin{aligned}
L_{\mathfrak{g}}^{\omega_1} &:= e^{\omega_1} + e^{-\omega_1} + e^{(-\omega_1+\omega_2)} + e^{(-\omega_2+\omega_3)} + e^{(\omega_2-\omega_3)} + e^{(\omega_1-\omega_2)};\\
L_{\mathfrak{g}}^{\omega_2} &:= e^{\omega_2} + e^{-\omega_2} + e^{(\omega_1-\omega_2+\omega_3)} + e^{(\omega_1+\omega_2-\omega_3)} + e^{(-\omega_1+\omega_3)} + e^{(2\omega_1-\omega_2)}\\
&\quad -e^{(-\omega_1+2\omega_2-\omega_3)} + 2 + e^{(\omega_1-2\omega_2+\omega_3)} + e^{(-2\omega_1+\omega_2)} + e^{(\omega_1-\omega_3)} + e^{(-\omega_1-\omega_2+\omega_3)}\\
&\quad +e^{(-\omega_1+\omega_2-\omega_3)};\\
L_{\mathfrak{g}}^{\omega_3} &:= e^{\omega_3} + e^{-\omega_3} + e^{(2\omega_2-\omega_3)} + e^{\omega_1} + e^{(2\omega_1-2\omega_2+\omega_3)} + e^{(-\omega_1+\omega_2)} + e^{(2\omega_1-\omega_3)} +\\
&\quad e^{(-\omega_2+\omega_3)} + e^{(\omega_2-\omega_3)} + e^{(-2\omega_1+\omega_3)} + e^{(\omega_1-\omega_2)} + e^{(-2\omega_1+2\omega_2-\omega_3)} + e^{-\omega_1}\\
&\quad +e^{(-2\omega_2+\omega_3)};
\end{aligned}
$$

C_3

C_3

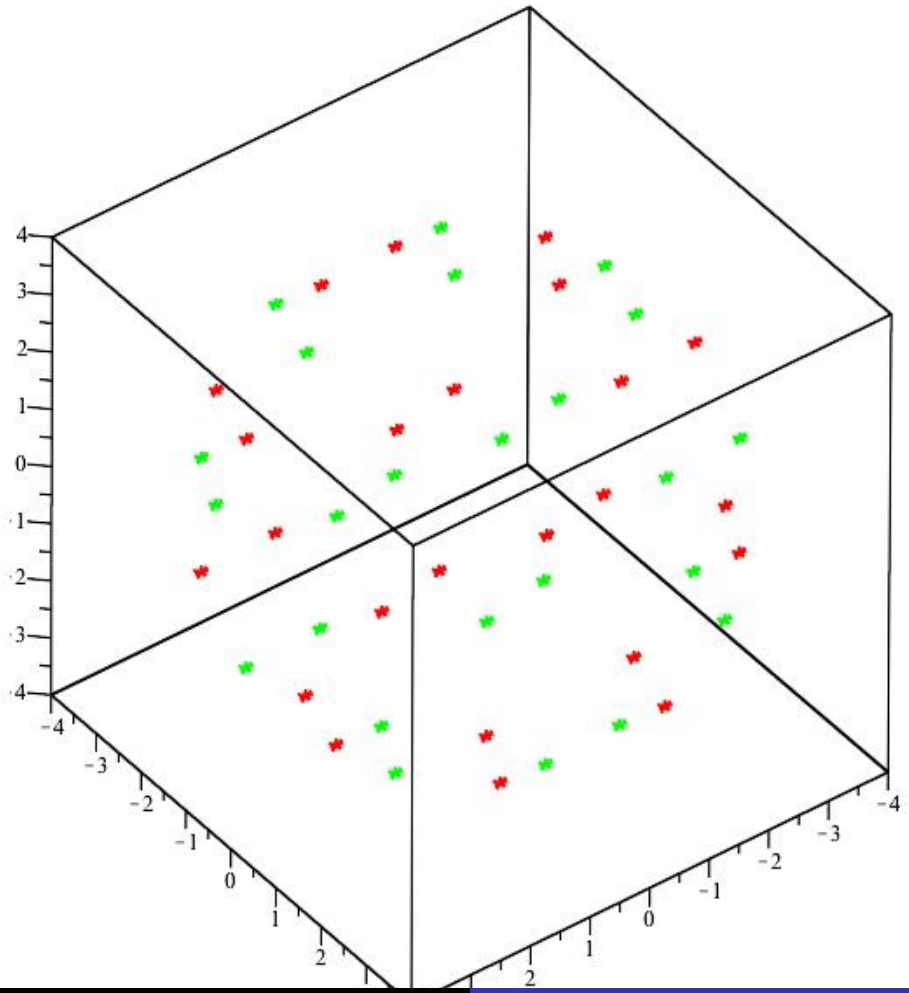


C_3

$$\begin{aligned}
\rho_{C_3} = \Phi^{(0)}_{C_3} &:= e^{(4\omega_1-3\omega_2+2\omega_3)} + e^{(\omega_1-2\omega_2+3\omega_3)} + e^{(-\omega_1+4\omega_2-\omega_3)} + e^{(2\omega_1+3\omega_2-2\omega_3)} \\
&- e^{(\omega_1+3\omega_2+\omega_3)} - e^{(2\omega_1-\omega_2+2\omega_3)} + e^{(\omega_1+\omega_2+\omega_3)} - e^{(-\omega_1+2\omega_2+\omega_3)} \\
&- e^{(\omega_1+4\omega_2-3\omega_3)} - e^{(-\omega_1-\omega_2+3\omega_3)} - e^{(-2\omega_1+5\omega_2-2\omega_3)} - e^{(3\omega_1-4\omega_2+3\omega_3)} \\
&+ e^{(-2\omega_1+\omega_2+2\omega_3)} - e^{(-5\omega_1+4\omega_2-\omega_3)} - e^{(\omega_1-5\omega_2+3\omega_3)} - e^{(-5\omega_1+\omega_2+\omega_3)} \\
&- e^{(4\omega_1-5\omega_2+2\omega_3)} - e^{(3\omega_1+\omega_2-3\omega_3)} - e^{(-3\omega_1-2\omega_2+3\omega_3)} - e^{(5\omega_1-2\omega_2-\omega_3)} \\
&- e^{(-3\omega_1+5\omega_2-3\omega_3)} + e^{(5\omega_1-\omega_2-\omega_3)} + e^{(-4\omega_1+5\omega_2-2\omega_3)} + e^{(5\omega_1-4\omega_2+\omega_3)} \\
&+ e^{(-\omega_1+5\omega_2-3\omega_3)} + e^{(3\omega_1-5\omega_2+3\omega_3)} + e^{(3\omega_1+2\omega_2-3\omega_3)} + e^{(-3\omega_1-\omega_2+3\omega_3)} \\
&- e^{(4\omega_1+\omega_2-2\omega_3)} - e^{(-4\omega_1+\omega_2+2\omega_3)} - e^{(-\omega_1-\omega_2-\omega_3)} + e^{(\omega_1-2\omega_2-\omega_3)} \\
&+ e^{(-\omega_1-3\omega_2+\omega_3)} + e^{(-2\omega_1+\omega_2-2\omega_3)} - e^{(-\omega_1+2\omega_2-3\omega_3)} - e^{(-4\omega_1+3\omega_2-2\omega_3)} \\
&- e^{(\omega_1-4\omega_2+\omega_3)} - e^{(2\omega_1-\omega_2-2\omega_3)} - e^{(-2\omega_1-3\omega_2+2\omega_3)} + e^{(\omega_1+\omega_2-3\omega_3)} \\
&+ e^{(-\omega_1-4\omega_2+3\omega_3)} + e^{(-4\omega_1-\omega_2+2\omega_3)} + e^{(4\omega_1-\omega_2-2\omega_3)} + e^{(-3\omega_1+4\omega_2-3\omega_3)} \\
&+ e^{(-5\omega_1+2\omega_2+\omega_3)} + e^{(-5\omega_1+3\omega_2-\omega_3)} + e^{(2\omega_1-5\omega_2+2\omega_3)} - e^{(5\omega_1-3\omega_2+\omega_3)}
\end{aligned}$$

C_3

C_3



C_3

Using the obtained we write down the important decomposition explicitly by mutiplication and its specialization $(X_1)^{p_1}(X_1)^{p_1}(X_r)^{p_r}$. Introduce a partial ordering by projecting the vector $\sum_i^3 p_i\omega_i$ to an appropriate "positive" test vector. This induces an ordering in the set of moments $\left\{X_{I(p)}\right\}$ in the set of Chebyshev polynomials. In terms of singular elements $\Phi^{(\omega_j)}$ we get the product

$$\int_0^1\int_0^1\int_0^1 (L_{\mathfrak{g}}^{\omega_1})^{\otimes p_1}\,(L_{\mathfrak{g}}^{\omega_2})^{\otimes p_2}\,(L_{\mathfrak{g}}{}^{\omega_3})^{\otimes p_3}\,\Phi^{(0)}e^{(-\omega_1-\omega_2-\omega_3)}|_{\overline{C^{(0)}}}dx^1\,dx^2\,dx^3$$
$$:= \quad M(m^{(\sum_i^3 p_i\omega_i)})$$

This $M(m^{(\sum_i^3 p_i\omega_i)})$ is a row. The function extracts the coefficient in the polynomial.

C_3

$$\int_0^1 \int_0^1 \int_0^1 \left(L_{\mathfrak{g}}^{\omega 1}\right)^{\otimes p_1} \left(L_{\mathfrak{g}}^{\omega 2}\right)^{\otimes p_2} \left(L_{\mathfrak{g}}{}^{\omega 3}\right)^{\otimes p_3} \Phi^{(0)} e^{-\rho} |\overline{C^{(0)}} e^{\sum_i^3 l_i \omega_i} dx^1 dx^2 dx^3$$

$$= \quad m^{\left(\sum_i^3 p_i \omega_i\right)\left(\sum_j^3 l_j \omega_j\right)}.$$

Collecting the terms with a fixed dominant weight $\mu \in P^{++}$ we obtain a positive integer coefficient $m^{\left(\sum_i^3 p_i \omega_i\right)\left(\sum_j^3 l_j \omega_j\right)}$. These coefficients form a triangular matrix M that has units in its main diagonal. The inverse matrix M^{-1} gives us explicit expressions for Chebyshev polynomials,

$$U^{H(p)} = \left(M^{-1}\right)^{H(p)}_{I(p)} X_{I(p)}.$$

We are able to write the concrete Chebyshev polynomials.

C_3

$$
\begin{aligned}
U_{0,0,0} &:= 1 \\
U_{1,0,0} &:= X1 \\
U_{0,1,0} &:= X2 \\
U_{0,0,1} &:= X3 \\
U_{2,0,0} &:= -1 - X2 + X1^2 \\
U_{1,1,0} &:= X1\,X2 - X1 - X3 \\
U_{1,0,1} &:= -X2 + X1\,X3 \\
U_{0,2,0} &:= X2 - X1^2 + X2^2 - X1\,X3 \\
U_{3,0,0} &:= -2\,X1\,X2 - X1 + X3 + X1^3 \\
U_{0,1,1} &:= -X1\,X2 + X3 + X2\,X3 \\
U_{2,1,0} &:= 1 - X1^2 - X2^2 - X1\,X3 + X1^2\,X2 \\
U_{2,0,1} &:= -X1\,X2 + X1 - X3 - X2\,X3 + X1^2\,X3 \\
U_{0,0,2} &:= -X2^2 + X3^2 + X1\,X3 \\
U_{1,2,0} &:= X1\,X2 + X1 - X1^3 - X2\,X3 - X1^2\,X3 + X1\,X2^2 \\
U_{1,1,1} &:= X1^2 - X3^2 - X1^2\,X2 + X1\,X2\,X3 \\
U_{4,0,0} &:= X2 - X1^2 + X2^2 + X1^4 + 2\,X1\,X3 - 3\,X1^2\,X2 \\
U_{3,1,0} &:= X1\,X2 + X1 + X3 - X1^3 + 2\,X2\,X3 - X1^2\,X3 - 2\,X1\,X2^2 + X1^3\,X2 \\
U_{1,0,2} &:= X1\,X2 - X3 - X2\,X3 + X1^2\,X3 - X1\,X2^2 + X1\,X3^2 \\
U_{3,0,1} &:= -1 + X2 + X1^2 + X2^2 + X3^2 - X1\,X3 - X1^2\,X2 + X1^3\,X3 - 2\,X1\,X2\,X3 \\
U_{0,3,0} &:= -1 - X2 + X1^2 + X2^2 + X3^2 + X2^3 + X1\,X3 - X1^2\,X2 - 2\,X1\,X2\,X3 \\
U_{0,2,1} &:= -X1\,X2 - X1 + X3 + X1^3 + 3\,X2\,X3 - X1^2\,X3 - X1\,X2^2 - X1\,X3^2 + X2^2\,X3 \\
U_{2,2,0} &:= X1^2 - X2^2 - X1^4 - X2^3 + X1\,X3 + 2\,X1^2\,X2 - X1^3\,X3 + X1^2\,X2^2 \\
U_{2,1,1} &:= X1\,X2 - X1 + X1^2\,X2\,X3 + X1^3 - X2\,X3 + X1\,X2^2 - X1^3\,X2 - X1\,X3^2 \\
&\quad - X2^2\,X3
\end{aligned}
$$

C_3

$$
\begin{aligned}
U_{1,1,1} &:= X1^2 - X3^2 - X1^2\,X2 + X1\,X2\,X3 \\
U_{4,0,0} &:= X2 - X1^2 + X2^2 + X1^4 + 2\,X1\,X3 - 3\,X1^2\,X2 \\
U_{3,1,0} &:= X1\,X2 + X1 + X3 - X1^3 + 2\,X2\,X3 - X1^2\,X3 - 2\,X1\,X2^2 + X1^3\,X2 \\
U_{1,0,2} &:= X1\,X2 - X3 - X2\,X3 + X1^2\,X3 - X1\,X2^2 + X1\,X3^2 \\
U_{3,0,1} &:= -1 + X2 + X1^2 + X2^2 + X3^2 - X1\,X3 - X1^2\,X2 + X1^3\,X3 - 2\,X1\,X2\,X3 \\
U_{0,3,0} &:= -1 - X2 + X1^2 + X2^2 + X3^2 + X2^3 + X1\,X3 - X1^2\,X2 - 2\,X1\,X2\,X3 \\
U_{0,2,1} &:= -X1\,X2 - X1 + X3 + X1^3 + 3\,X2\,X3 - X1^2\,X3 - X1\,X2^2 - X1\,X3^2 + X2^2\,X3 \\
U_{2,2,0} &:= X1^2 - X2^2 - X1^4 - X2^3 + X1\,X3 + 2\,X1^2\,X2 - X1^3\,X3 + X1^2\,X2^2 \\
U_{2,1,1} &:= X1\,X2 - X1 + X1^2\,X2\,X3 + X1^3 - X2\,X3 + X1\,X2^2 - X1^3\,X2 - X1\,X3^2 \\
&\quad -X2^2\,X3 \\
U_{0,1,2} &:= X2 - X1^2 + X3^2 - X2^3 - X1\,X3 + X1^2\,X2 + X2\,X3^2 \\
U_{2,0,2} &:= -X2 + X2^2 - X3^2 + X2^3 - X1\,X3 + X1^2\,X2 + X1^3\,X3 - X1^2\,X2^2 - X2\,X3^2 \\
&\quad +X1^2\,X3^2 - 2\,X1\,X2\,X3 \\
U_{5,0,0} &:= 2\,X1\,X2 - 2\,X3 - X1^3 + X1^5 - 2\,X2\,X3 + 3\,X1^2\,X3 + 3\,X1\,X2^2 - 4\,X1^3\,X2
\end{aligned}
$$

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-

C_3

$$
\begin{aligned}
U_{1,3,0} &:= -X1\,X2 - X1 - X3 - 2\,X1^2\,X2\,X3 + X1^3 - 2\,X2\,X3 + 3\,X1^2\,X3 + X1\,X2^2 \\
&\quad -X1^3\,X2 + 2\,X1\,X3^2 - X2^2\,X3 + X1\,X2^3 \\
U_{4,1,0} &:= -2\,X2 + X1^2 - X3^2 - X1^4 + X2^3 + 2\,X1^2\,X2 - X1^3\,X3 - 3\,X1^2\,X2^2 + X1^4\,X2 \\
&\quad +4\,X1\,X2\,X3 \\
U_{1,2,1} &:= 1 + X1\,X2^2\,X3 - 2\,X1^2 - X2^2 - X3^2 + X1^4 + X1\,X3 - X1^3\,X3 - X1^2\,X2^2 \\
&\quad -X2\,X3^2 - X1^2\,X3^2 + 4\,X1\,X2\,X3 \\
U_{4,0,1} &:= -2\,X1 - 3\,X1^2\,X2\,X3 + X1^3 - X1^2\,X3 + 2\,X1\,X2^2 - X1^3\,X2 + 2\,X1\,X3^2 \\
&\quad +X2^2\,X3 + X1^4\,X3 \\
U_{0,0,3} &:= 2\,X1\,X2 - X3 - X1^3 + X3^3 - 2\,X2\,X3 + X1\,X2^2 + 2\,X1\,X3^2 - 2\,X2^2\,X3 \\
U_{3,2,0} &:= -2\,X1\,X2 + X1^2\,X2\,X3 + X1^3 - X1^5 + 2\,X2\,X3 - 2\,X1\,X2^2 + 3\,X1^3\,X2 \\
&\quad -X1\,X3^2 + 2\,X2^2\,X3 - 2\,X1\,X2^3 - X1^4\,X3 + X1^3\,X2^2 \\
U_{1,1,2} &:= -X1\,X2 + X1 + X3 - X1^3 - X3^3 - X1^2\,X3 + X1\,X2^2 + X1^3\,X2 - X1\,X3^2 \\
&\quad +X2^2\,X3 - X1\,X2^3 + X1\,X2\,X3^2 \\
U_{3,1,1} &:= X1^3\,X2\,X3 - 2\,X1\,X2^2\,X3 - 2\,X1^2 + 2\,X3^2 + X1^4 + 2\,X1^2\,X2^2 \\
&\quad +2\,X2\,X3^2 - X1^2\,X3^2 - X1^4\,X2 - 2\,X1\,X2\,X3 \\
U_{0,4,0} &:= -X2 - 3\,X1\,X2^2\,X3 - X2^2 + X3^2 + X2^3 + X2^4 - 2\,X1\,X3 + X1^2\,X2 \\
&\quad +2\,X1^3\,X3 - X1^2\,X2^2 + 2\,X2\,X3^2 + X1^2\,X3^2 - 2\,X1\,X2\,X3 \\
U_{3,0,2} &:= -2\,X1\,X2 - 3\,X1^2\,X2\,X3 + X3^3 + 2\,X2\,X3 - X1^2\,X3 + X1^3\,X2 + 2\,X1\,X2^3 \\
&\quad +X1^4\,X3 - X1^3\,X2^2 + X1^3\,X3^2 - 2\,X1\,X2\,X3^2 \\
U_{0,3,1} &:= X1 - X3 - X1^3 + X3^3 + 2\,X2\,X3 - X1^2\,X3 - 3\,X1\,X2^2 + 2\,X1^3\,X2 \\
&\quad +3\,X2^2\,X3 - X1\,X2^3 + X2^3\,X3 - 2\,X1\,X2\,X3^2 \\
U_{2,3,0} &:= 2\,X2 - 2\,X1^3\,X2\,X3 + X1\,X2^2\,X3 - X1^2 + 2\,X2^2 - X3^2 + X1^4 - X2^3 \\
&\quad -X2^4 - 2\,X1\,X3 - 3\,X1^2\,X2 + 3\,X1^3\,X3 + 2\,X1^2\,X2^2 - X2\,X3^2
\end{aligned}
$$

C_3

We attribute the multivariate Chebyshev function $X_{i_1}^{p_1}, X_{i_2}^{p_2}, X_{i_3}^{p_3}$ defined as a specialized character of type x:

$$\begin{aligned}\rho \quad := \quad & 512 + 298X_2^2 + 95X_1^4 + 59X_1^2 - 32X_2^4 + 256X_2 + X_1^2X_2^2X_3^2 + 18X_1X_2X_3^3 + \\ & + 86X_1^2X_2X_3^2 + 94X_1^3X_2X_3 - 96X_1X_2^3X_3 - 216X_1X_2^2X_3 - 738X_1X_2X_3 + \\ & + 22X_1^3X_2^2X_3 - 12X_1^5X_3 - 7X_1^4X_2^2 - 20X_1^4X_3^2 - 122X_1^2X_3^2 - 144X_1X_3^3 + \\ & + 384X_2X_3^2 + 36X_1^2X_2^2 + 132X_2^2X_3^2 - 4X_1^2X_2^4 + 15X_1^6 + 16X_2^5 - 4X_2^3X_3^2 - \\ & - 4X_1^3X_3^3 - 27X_3^4 + 220X_1^3X_3 - 157X_1^4X_2 - 414X_1^2X_2 + 374X_1^2X_2^2 + \\ & + 30X_1X_3 + 443X_3^2 - 107X_2^3\end{aligned}$$

C_3

The properties of singular elements,

$$\gamma_{\mathfrak{g}}(X) = \sqrt{b_{l(p)} X^{l(p)}} \, \frac{q_{\mathfrak{g}} (-1)^{\mathrm{maxlength(W)}}}{(2\pi i)^r \, |F_{\mathfrak{g}}| \, |W_{\mathfrak{g}}|}. \tag{9}$$

This is all of Chebyshev polynomials for C_3.

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